二重振り子の運動方程式

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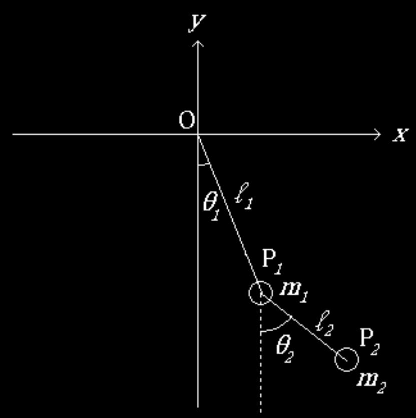

質点 1 の位置ベクトル

$$\boldsymbol{R}_1 = (x_1, y_1) = (l_1 \sin\theta_1, -l_1\cos\theta_1)$$

質点 2 の位置ベクトル

$$\boldsymbol{R}_2 = (x_2, y_2) = (l_1 \sin\theta_1 + l_2\sin\theta_2, -l_1\cos\theta_1 - l_2\cos\theta_2)$$

質点 1 の速度ベクトル

$$\boldsymbol{v}_1 = \frac{d\boldsymbol{R}_1}{dt} = \left(\frac{dx_1}{dt}, \frac{dy_1}{dt}\right) = (l_1\dot{\theta}_1\cos\theta_1, l_1\dot{\theta}_1\sin\theta_1)$$

質点 2 の速度ベクトル

$$\boldsymbol{v}_2 = \frac{d\boldsymbol{R}_2}{dt} = \left(\frac{dx_2}{dt}, \frac{dy_2}{dt}\right) = (l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2, l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2)$$

全位置エネルギー

$$\begin{aligned} U &= -m_1 g l_1\cos\theta_1 - m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2) \\ &= -(m_1+m_2)gl_1\cos\theta_1 - m_2 g l_2\cos\theta_2 \end{aligned}$$

全運動エネルギー

$$\begin{aligned} K &= \frac{1}{2}m_1|\boldsymbol{v}_1|^2 + \frac{1}{2}m_2|\boldsymbol{v}_2|^2 \\ &= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] \\ &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \end{aligned}$$

ラグランジアン

$$\begin{aligned} L &= K - U \\ &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + (m_1+m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2 \end{aligned}$$

$\omega_1 := \dot{\theta}_1, \omega_2 := \dot{\theta}_2$ とすれば

$$L = \frac{1}{2}(m_1+m_2)l_1^2\omega_1^2 + \frac{1}{2}m_2l_2^2\omega_2^2 + m_2l_1l_2\omega_1\omega_2\cos(\theta_1-\theta_2) + (m_1+m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2$$

質点 1 の運動方程式

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \tag{1}$$

質点 2 の運動方程式

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \tag{2}$$

ここで

$$\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= -m_2 l_1 l_2 \omega_1 \omega_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1 \\
\frac{\partial L}{\partial \theta_2} &= m_2 l_1 l_2 \omega_1 \omega_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 \\
\frac{\partial L}{\partial \omega_1} &= (m_1 + m_2) l_1^2 \omega_1 + m_2 l_1 l_2 \omega_2 \cos(\theta_1 - \theta_2) \\
\frac{\partial L}{\partial \omega_2} &= m_2 l_2^2 \omega_2 + m_2 l_1 l_2 \omega_1 \cos(\theta_1 - \theta_2)
\end{aligned}$$

である。

(1) を計算する。

$$\begin{aligned}
&\frac{d}{dt}\left[(m_1 + m_2) l_1^2 \omega_1 + m_2 l_1 l_2 \omega_2 \cos(\theta_1 - \theta_2)\right] + m_2 l_1 l_2 \omega_1 \omega_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0 \\
\therefore\ &(m_1 + m_2) l_1^2 \dot{\omega}_1 + m_2 l_1 l_2 \dot{\omega}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \omega_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \omega_1 \omega_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0 \\
\therefore\ &(m_1 + m_2) l_1^2 \dot{\omega}_1 + m_2 l_1 l_2 \dot{\omega}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \omega_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0
\end{aligned}$$

$\alpha_1 := \dot{\omega}_1, \alpha_2 := \dot{\omega}_2$ とすれば

$$(m_1 + m_2) l_1^2 \alpha_1 + m_2 l_1 l_2 \alpha_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \omega_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0 \tag{3}$$

(2) を計算する。

$$\frac{d}{dt}\left[m_2 l_2^2 \omega_2 + m_2 l_1 l_2 \omega_1 \cos(\theta_1 - \theta_2)\right] - m_2 l_1 l_2 \omega_1 \omega_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

$$\therefore\ m_2 l_2^2 \alpha_2 + m_2 l_1 l_2 \alpha_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \omega_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0 \tag{4}$$

(3),(4) を α_1, α_2 に関する二元連立一次方程式として解く。

$$\begin{aligned}
c_{11} &= (m_1 + m_2) l_1^2 \\
c_{12} &= m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \\
c_{21} &= m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \\
c_{22} &= m_2 l_2^2 \\
b_1 &= -m_2 l_1 l_2 \omega_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1 \\
b_2 &= m_2 l_1 l_2 \omega_1^2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2
\end{aligned}$$

とすれば

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

クラーメルの公式を用いて

$$\alpha_1 = \frac{\begin{vmatrix} b_1 & c_{12} \\ b_2 & c_{22} \end{vmatrix}}{\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}},\ \alpha_2 = \frac{\begin{vmatrix} c_{11} & b_1 \\ c_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}}$$

ここで

$$\begin{aligned}
\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} &= c_{11} c_{22} - c_{12} c_{21} \\
&= (m_1 + m_2) l_1^2 m_2 l_2^2 - m_2^2 l_1^2 l_2^2 \cos^2(\theta_1 - \theta_2)
\end{aligned}$$

$$(5)$$

$$\begin{vmatrix} b_1 & c_{12} \\ b_2 & c_{22} \end{vmatrix} = b_1 c_{22} - c_{12} b_2$$

$$= -\left[m_2 l_1 l_2 \omega_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1\right] m_2 l_2^2$$

$$-m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\left[m_2 l_1 l_2 \omega_1^2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2\right]$$

$$(6)$$

$$\begin{vmatrix} c_{11} & b_1 \\ c_{21} & b_2 \end{vmatrix} = c_{11} b_2 - b_1 c_{21}$$

$$= (m_1 + m_2) l_1^2 \left[m_2 l_1 l_2 \omega_1^2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2\right]$$

$$+\left[m_2 l_1 l_2 \omega_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1\right] m_2 l_1 l_2 \cos(\theta_1 - \theta_2)$$

$$(7)$$

(5),(6),(7) を見て分かるように、α_1, α_2 の分母, 分子は $m_1 + m_2$ または m_2 或いはその積で括れる部分が多い。そこで $\eta = \frac{m_1 + m_2}{m_2}$ とし、さらに $\phi = \theta_1 - \theta_2$ とすると

$$m_1 + m_2 = m_2 \eta,\ m_2(m_1 + m_2) = m_2^2 \eta$$

であることに注意して

$$(5) = m_2^2 l_1^2 l_2^2 \left(\frac{m_1 + m_2}{m_2} - \cos^2\phi\right)$$

$$= m_2^2 l_1^2 l_2^2 \left(\eta - \cos^2\phi\right)$$

$$(8)$$

$$(6) = -m_2^2 l_1 l_2^3 \omega_2^2 \sin\phi - m_2^2 \eta g l_1 l_2^2 \sin\theta_1 - m_2^2 l_1^2 l_2^2 \omega_1^2 \sin\phi\cos\phi + m_2^2 g l_1 l_2^2 \sin\theta_2 \cos\phi \qquad (9)$$

$$(7) = m_2^2 \eta l_1^3 l_2 \omega_1^2 \sin\phi - m_2^2 \eta g l_1^2 l_2 \sin\theta_2 + m_2^2 l_1^2 l_2^2 \omega_2^2 \sin\phi\cos\phi + m_2^2 \eta g l_1^2 l_2 \sin\theta_1 \cos\phi \qquad (10)$$

と簡単になる。これらを用いて

$$\alpha_1 = \frac{(9)}{(8)}$$

$$= \frac{-l_2 \omega_2^2 \sin\phi - \eta g \sin\theta_1 - l_1 \omega_1^2 \sin\phi\cos\phi + g\cos\phi\sin\theta_2}{l_1(\eta - \cos^2\phi)}$$

$$= \frac{g(\cos\phi\sin\theta_2 - \eta\sin\theta_1) - \left(l_1\omega_1^2\cos\phi + l_2\omega_2^2\right)\sin\phi}{l_1(\eta - \cos^2\phi)}$$

$$\alpha_2 = \frac{(10)}{(8)}$$

$$= \frac{\eta l_1 \omega_1^2 \sin\phi - \eta g \sin\theta_2 + l_2 \omega_2^2 \sin\phi\cos\phi + \eta g \sin\theta_1 \cos\phi}{l_2(\eta - \cos^2\phi)}$$

$$= \frac{g\eta(\cos\phi\sin\theta_1 - \sin\theta_2) + \left(\eta l_1 \omega_1^2 + l_2 \omega_2^2 \cos\phi\right)\sin\phi}{l_2(\eta - \cos^2\phi)}$$

以上。